Digital Communication Systems EES 452

Asst. Prof. Dr. Prapun Suksompong

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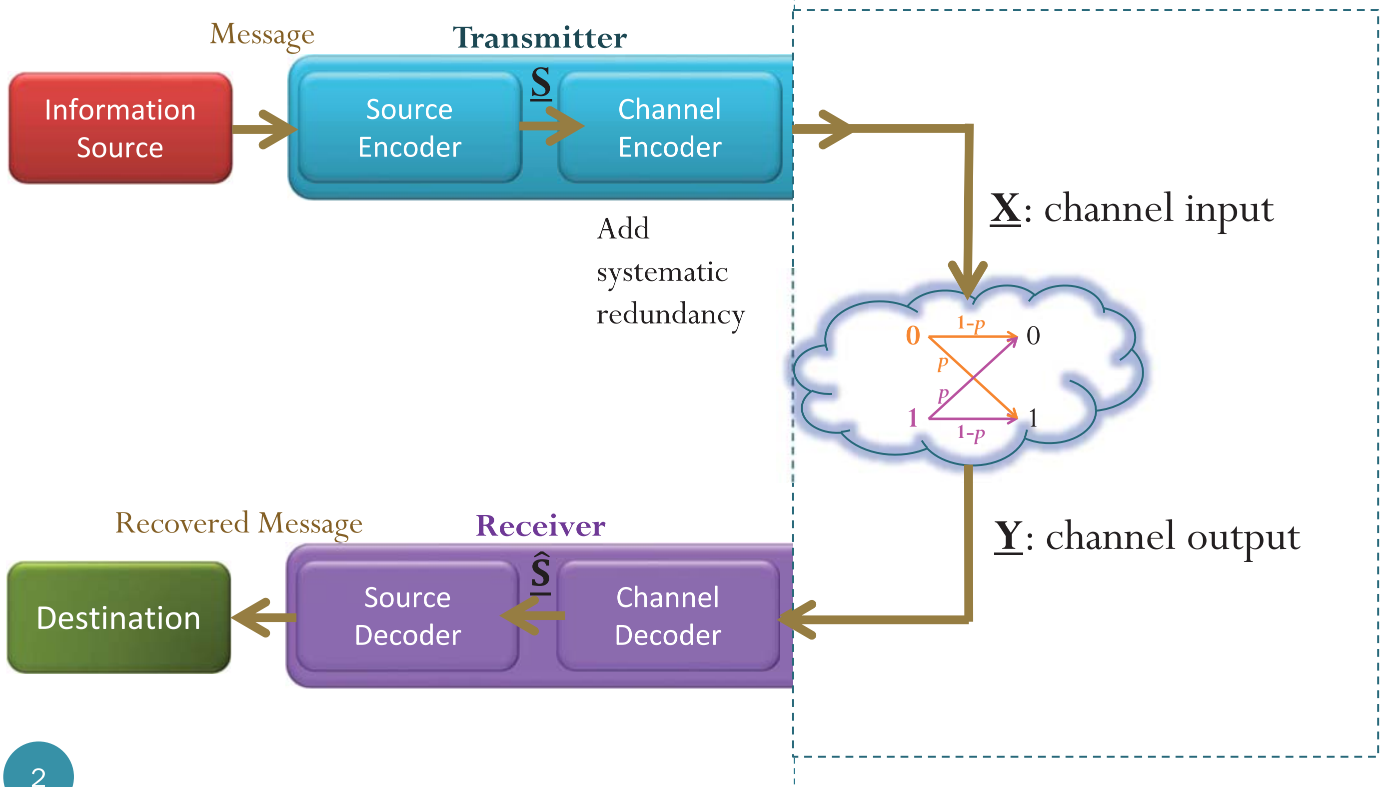
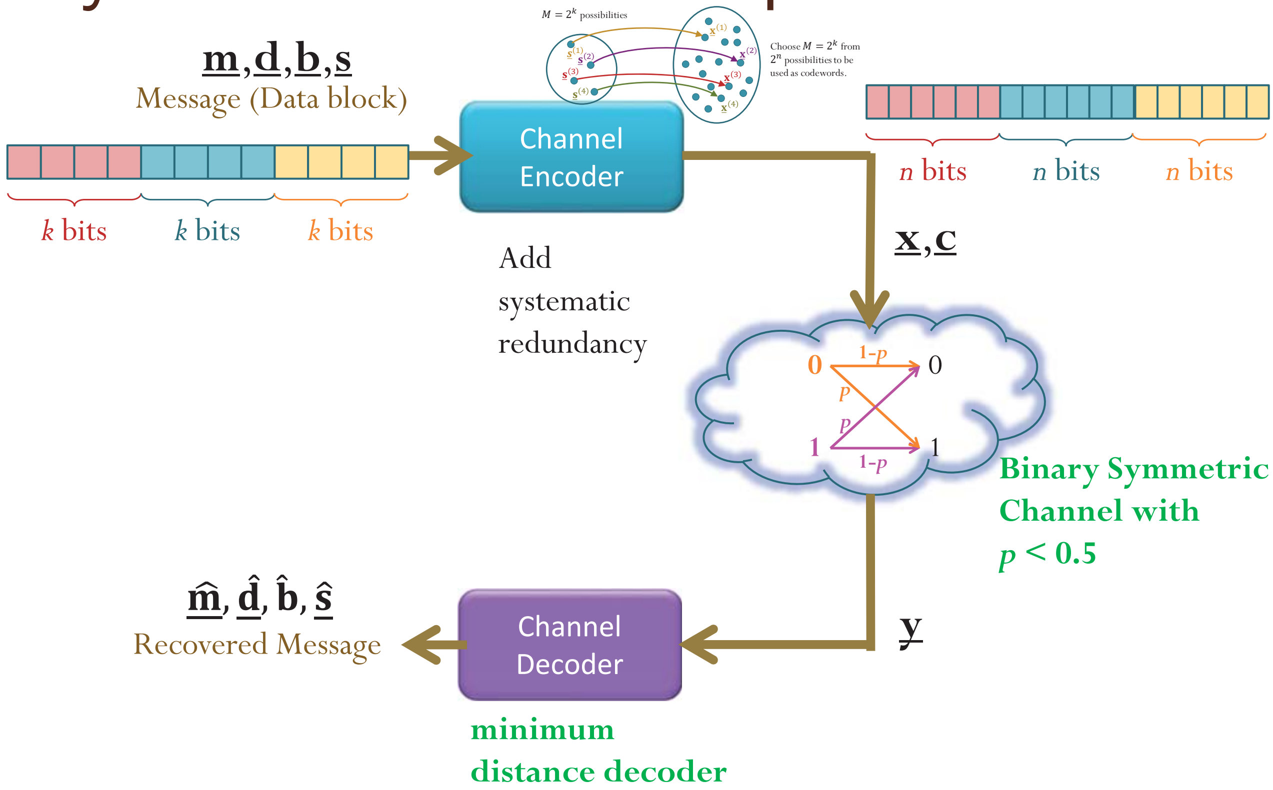
5. Channel Coding

Review: Channel Encoder and Decoder



System Model for Chapter 5



Vector Notation

- $\vec{\mathbf{v}}$: column vector $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}$
- $\underline{\mathbf{r}}$: row vector $(r_1, r_2, \ldots, r_i, \ldots r_n)$
- Subscripts represent element indices inside individual vectors.
  - v_i and r_i refer to the i^{th} elements inside the vectors $\vec{\mathbf{v}}$ and $\underline{\mathbf{r}}$, respectively.
- When we have a list of vectors, we use superscripts in parentheses as indices of vectors.
  - $\vec{\mathbf{v}}^{(1)}, \vec{\mathbf{v}}^{(2)}, \ldots, \vec{\mathbf{v}}^{(M)}$ is a list of M column vectors
  - $\underline{\mathbf{r}}^{(1)}, \underline{\mathbf{r}}^{(2)}, \ldots, \underline{\mathbf{r}}^{(M)}$ is a list of M row vectors
  - $\vec{\mathbf{v}}^{(i)}$ and $\underline{\mathbf{r}}^{(i)}$ refer to the i^{th} vectors in the corresponding lists.

$\vec{\mathbf{0}}, \underline{\mathbf{0}}$: the zero vector (the all-zero vector)

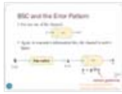
$\vec{\mathbf{1}}, \underline{\mathbf{1}}$: the one vector (the all-one vector)

Channel Decoding

- Recall

MAP decoder is optimal $\xrightarrow{\text{Codewords are equally likely}}$ ML decoder is optimal $\xrightarrow{\text{BSC with } p < 0.5}$ Min distance decoder is optimal

1. **MAP decoder** is the optimal decoder.
2. When the codewords are equally-likely, the **ML decoder** the same as the MAP decoder; hence it is also **optimal**.
3. When the **crossover probability** of the BSC p is < 0.5,
   ML decoder is the same as the **minimum distance decoder**.



- In this chapter, we assume the use of **minimum distance decoder**.
  - $\hat{\underline{\mathbf{x}}}(\underline{\mathbf{y}}) = \arg\min_{\underline{\mathbf{x}}} d(\underline{\mathbf{x}}, \underline{\mathbf{y}})$
- Also, in this chapter, we will focus
  - less on probabilistic analysis,
  - but more on explicit codes.

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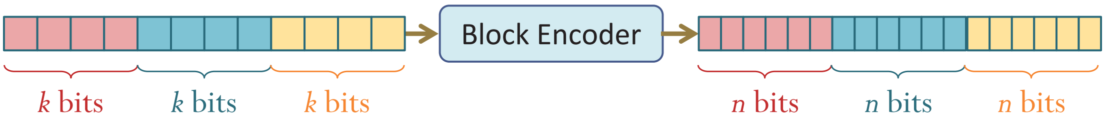
5.1 Binary Linear Block Codes

Review: Block Encoding

- We mentioned the general form of channel coding over BSC.
- In particular, we looked at the general form of **block codes**.



Code length

"Dimension" of the code

- **(n,k) codes**: n-bit blocks are used to conveys k-info-bit blocks

codewords "messages"

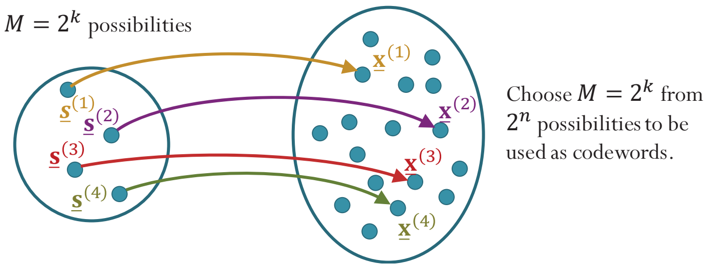
- **Assume $n > k$**
- **Rate**: $R = \frac{k}{n}$.

Max. achievable rate

> Recall that the capacity of BSC is $C = 1 - H(p)$.
> For $p \in (0,1)$, we also have $C \in (0,1)$.
> Achievable rate is < 1.

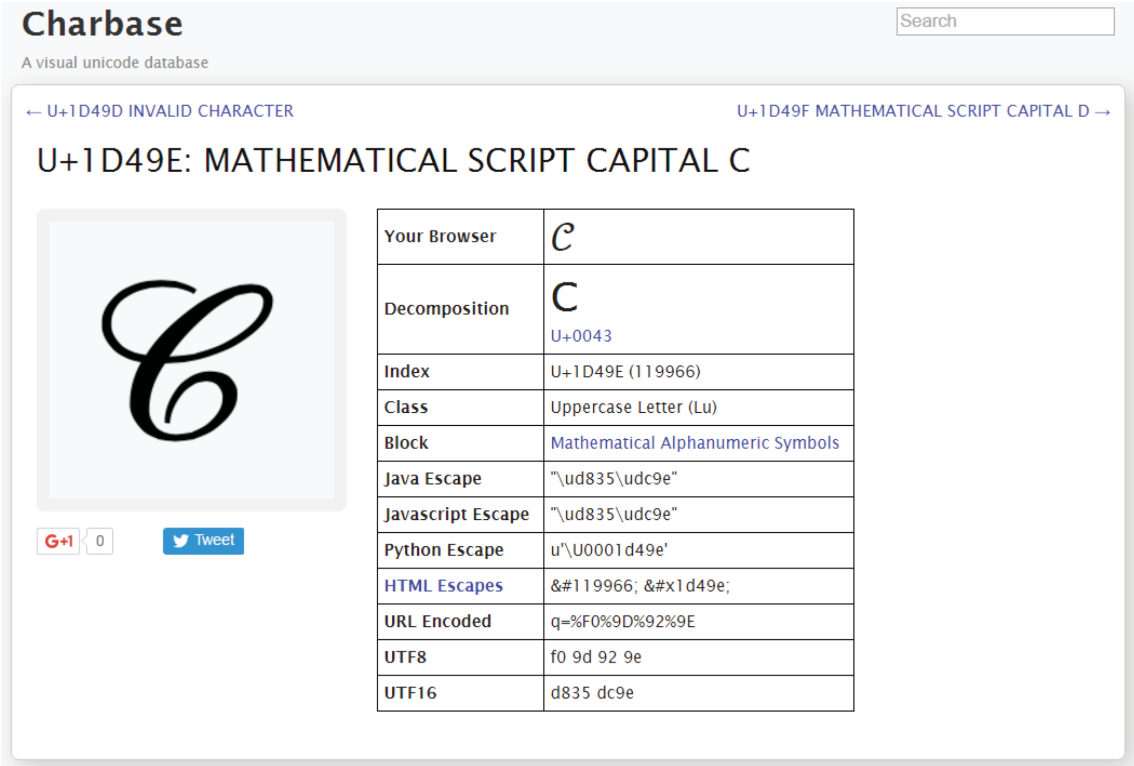
$\mathcal{C}$

- $\mathcal{C}$ = the collection of all codewords for the code considered.
- Each n-bit block is selected from $\mathcal{C}$.
- The message (data block) has k bits, so there are 2^k possibilities.
- A reasonable code would not assign
  the same codeword to different messages.
- Therefore, there are 2^k (distinct) codewords in $\mathcal{C}$.



- Ex. Repetition code with $n = 3$

MATHEMATICAL SCRIPT CAPITAL C



Charbase
A visual unicode database

← U+1D49D INVALID CHARACTER — U+1D49F MATHEMATICAL SCRIPT CAPITAL D →

U+1D49E: MATHEMATICAL SCRIPT CAPITAL C

Your Browser	𝒞
Decomposition	C U+0043
Index	U+1D49E (119966)
Class	Uppercase Letter (Lu)
Block	Mathematical Alphanumeric Symbols
Java Escape	"\ud835\udc9e"
Javascript Escape	"\ud835\udc9e"
Python Escape	u'\U0001d49e'
HTML Escapes	𝒞 𝒞
URL Encoded	q=%F0%9D%92%9E
UTF8	f0 9d 92 9e
UTF16	d835 dc9e

[http://www.charbase.com/1d49e-unicode-mathematical-script-capital-c]

GF(2)

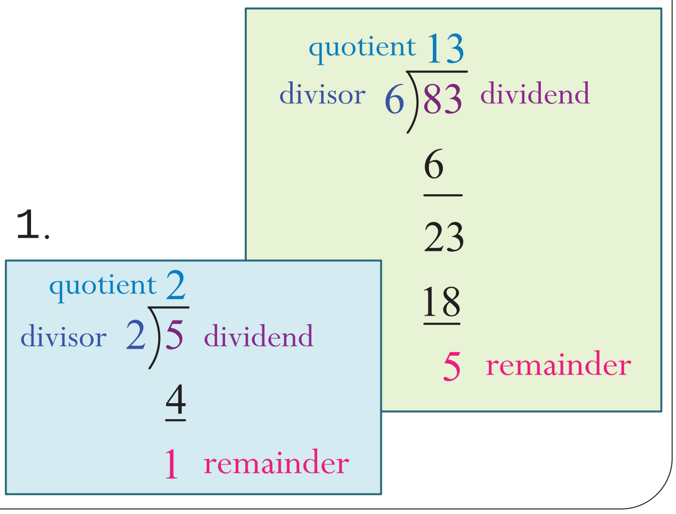
- The construction of the codes can be expressed in matrix form using the following definition of **addition** and **multiplication** of bits:

$$\begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \bullet & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

- These are **modulo-2** addition and **modulo-2** multiplication, respectively.
- The operations are the same as the **exclusive-or** (**XOR**) operation and the **AND** operation.
  - We will simply call them addition and multiplication so that we can use a matrix formalism to define the code.
- The two-element set {0, 1} together with this definition of addition and multiplication is a number system called a **finite field** or a **Galois field**, and is denoted by the label **GF(2)**.

Modulo operation

- The **modulo operation** finds the **remainder** after division of one number by another (sometimes called **modulus**).
- Given two positive numbers, a (the **dividend**) and n (the **divisor**),
- $\boldsymbol{a}$ **modulo** $\boldsymbol{n}$ (abbreviated as $\boldsymbol{a}$ **mod** $\boldsymbol{n}$) is the remainder of the division of a by n.
- "83 mod 6" = 5
- "5 mod 2" = 1
  - In MATLAB, `mod(5,2) = 1`.
- **Congruence relation**
  - $5 \equiv 1 \pmod 2$



GF(2) and modulo operation

- Normal addition and multiplication (for 0 and 1):

+	0	1
0	0	1
1	1	2

×	0	1
0	0	0
1	0	1

- Addition and multiplication in GF(2):

⊕	0	1
0	0	1
1	1	0

•	0	1
0	0	0
1	0	1

GF(2)

- The construction of the codes can be expressed in matrix form using the following definition of addition and multiplication of bits:

$\oplus$	0	1
0	0	1
1	1	0

$\bullet$	0	1
0	0	0
1	0	1

- Note that

$$x \oplus 0 = x$$ $0 \oplus 0 = 0$, $1 \oplus 0 = 1$

$$x \oplus 1 = \bar{x}$$ $0 \oplus 1 = 1$, $1 \oplus 1 = 0$

$$x \oplus x = 0$$ $0 \oplus 0 = 0$, $1 \oplus 1 = 0$

The property above implies $\underbrace{-x}_{} = x$

By definition, "$-x$" is something that, when added with x, gives 0.

- Extension: For vector and matrix, apply the operations to the elements the same way that addition and multiplication would normally apply (except that the calculations are all in GF(2)).

Examples

- Normal vector addition:

$$\begin{array}{rl} & [1 \quad -1 \quad 2 \quad 1] \\ + & [-2 \quad 3 \quad 0 \quad 1] \\ \hline = & [-1 \quad 2 \quad 2 \quad 2] \end{array}$$

- Vector addition in GF(2):

$$\begin{array}{rl} & [1 \quad 0 \quad 1 \quad 1] \\ \oplus & [0 \quad 1 \quad 0 \quad 1] \\ \hline = & [1 \quad 1 \quad 1 \quad 0] \end{array}$$

Alternatively, one can also apply normal vector addition first, then apply "mod 2" to each element:

$$\begin{array}{rl} & [1 \quad 0 \quad 1 \quad 1] \\ + & [0 \quad 1 \quad 0 \quad 1] \\ \hline = & [1 \quad 1 \quad 1 \quad 2] \end{array}$$

$$\Big\downarrow \text{mod } 2$$

$$[1 \quad 1 \quad 1 \quad 0]$$

Examples

- Normal matrix multiplication:

$$\left(7 \times (-2)\right) + (4 \times 3) + \left(3 \times (-7)\right) = -14 + 12 + (-21)$$

$$\begin{bmatrix} 7 & 4 & 3 \\ 2 & 5 & 6 \\ 1 & 8 & 9 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 3 & -8 \\ -7 & 6 \end{bmatrix} = \begin{bmatrix} -23 & 14 \\ -31 & 4 \\ -41 & -6 \end{bmatrix}$$

- Matrix multiplication in GF(2):

$$(1 \cdot 1) \oplus (0 \cdot 0) \oplus (1 \cdot 1) = 1 \oplus 0 \oplus 1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Alternatively, one can also apply normal matrix multiplication first, then apply "mod 2" to each element:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \xrightarrow{\text{mod } 2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

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5.1 Binary Linear Block Codes

Évariste Galois

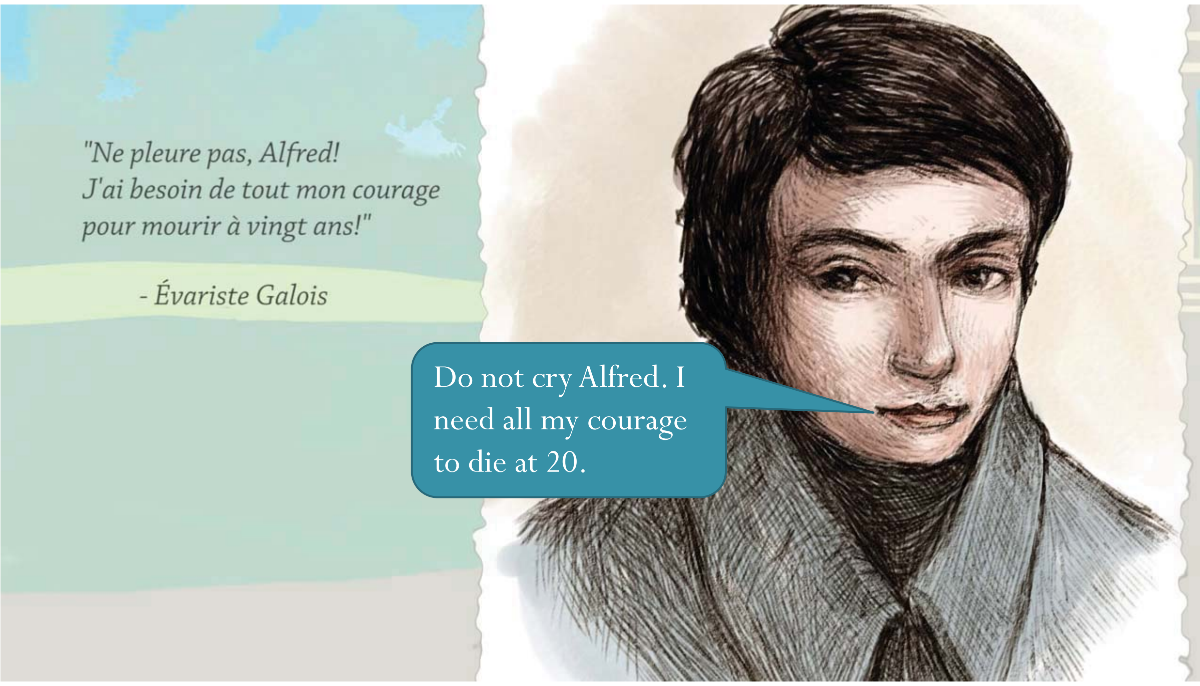
Évariste Galois

[https://www.youtube.com/watch?v=Mc0bvea6G3I]

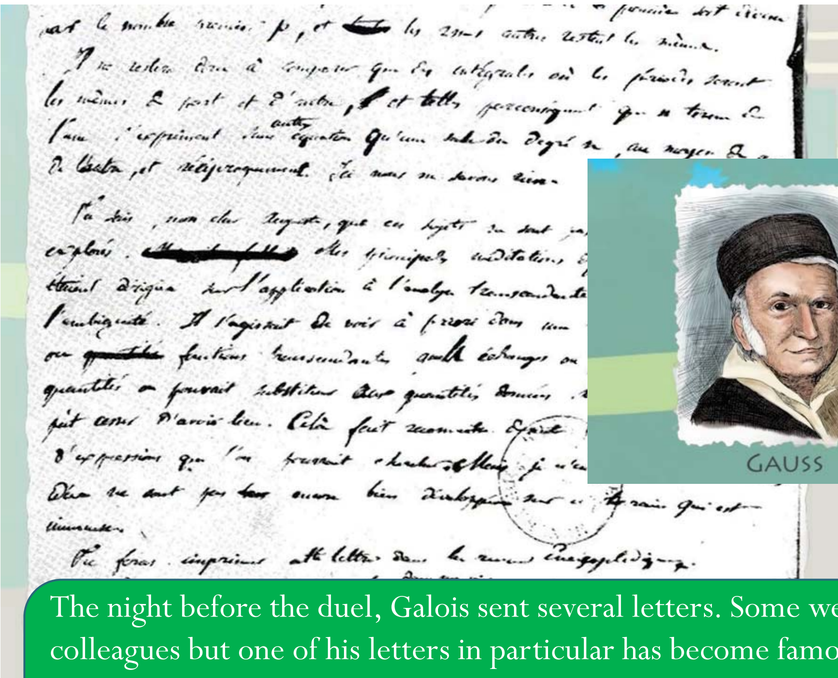
Évariste Galois

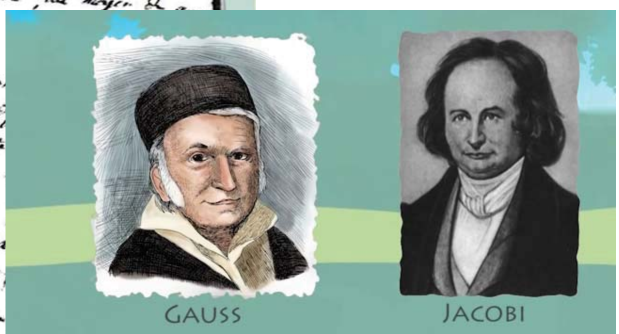
On the morning of May 30th, 1832, two men in Paris fought a duel. Not an unusual event for those days. One of the men was shot in the gut and died the following day…

Évariste Galois



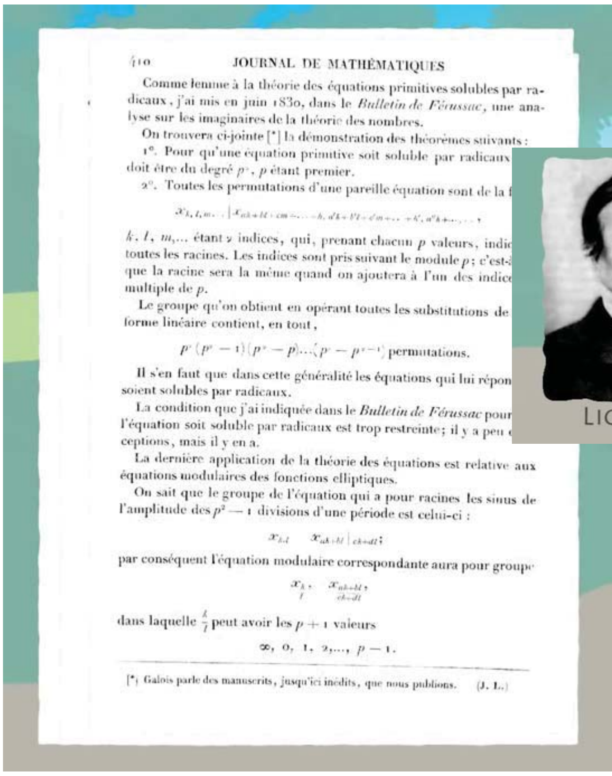
Évariste Galois





The night before the duel, Galois sent several letters. Some were to his political colleagues but one of his letters in particular has become famous amongst mathematicians. Fearing that he might die, Galois assembled his mathematical discoveries and sent them to his friend with instructions to pass him along to two of the best mathematicians of the day: Gauss and Jacobi.

Évariste Galois



410 JOURNAL DE MATHÉMATIQUES

Comme lemme à la théorie des équations primitives solubles par radicaux, j'ai mis en juin 1830, dans le *Bulletin de Férussac*, une analyse sur les imaginaires de la théorie des nombres.

On trouvera ci-jointe [*] la démonstration des théorèmes suivants :

1°. Pour qu'une équation primitive soit soluble par radicaux doit être du degré p^ν, p étant premier.

2°. Toutes les permutations d'une pareille équation sont de la f

$k, l, m,\ldots$ étant ν indices, qui, prenant chacun p valeurs, indi toutes les racines. Les indices sont pris suivant le module p; c'est-à que la racine sera la même quand on ajoutera à l'un des indice multiple de p.

Le groupe qu'on obtient en opérant toutes les substitutions de forme linéaire contient, en tout,

$$p^\nu(p^\nu - 1)(p^\nu - p)\ldots(p^\nu - p^{\nu-1}) \text{ permutations.}$$

Il s'en faut que dans cette généralité les équations qui lui répon soient solubles par radicaux.

La condition que j'ai indiquée dans le *Bulletin de Férussac* pour l'équation soit soluble par radicaux est trop restreinte; il y a peu ceptions, mais il y en a.

La dernière application de la théorie des équations est relative aux équations modulaires des fonctions elliptiques.

On sait que le groupe de l'équation qui a pour racines les sinus de l'amplitude des $p^2 - 1$ divisions d'une période est celui-ci :

$$x_{k,l} \qquad x_{ak+bl,\ ck+dl};$$

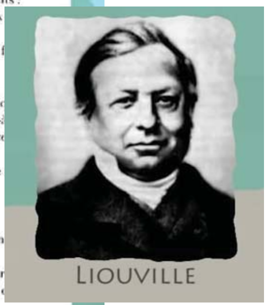
par conséquent l'équation modulaire correspondante aura pour groupe

$$x_{\frac{k}{l}}, \qquad x_{\frac{ak+bl}{ck+dl}},$$

dans laquelle $\frac{k}{l}$ peut avoir les $p+1$ valeurs

$$\infty,\ 0,\ 1,\ 2,\ldots,\ p-1.$$

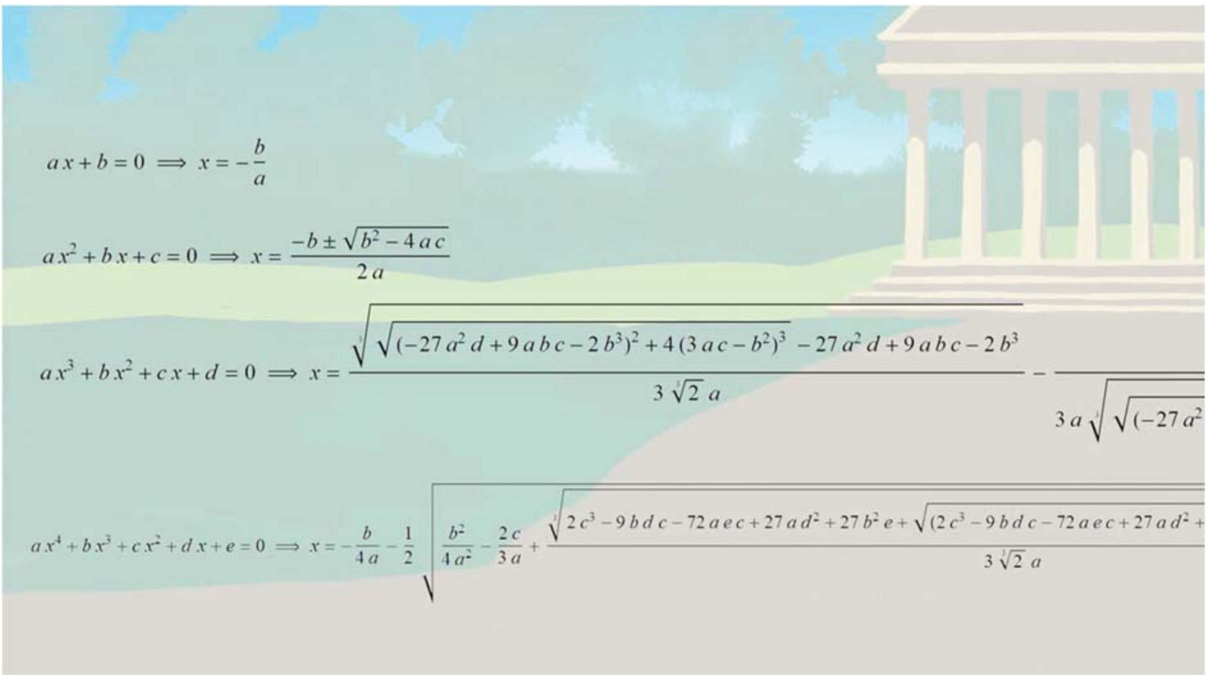
[*] Galois parle des manuscrits, jusqu'ici inédits, que nous publions. (J. L.)



The papers laid dormant until over a decade later when the letter made its way to the mathematician Liouville who took the time to read through the manuscripts and sought to their publication.

The world finally learned that as a teenager Galois had solved one of the most important problems in algebra.

Évariste Galois: Contribution



$$a x + b = 0 \implies x = -\frac{b}{a}$$

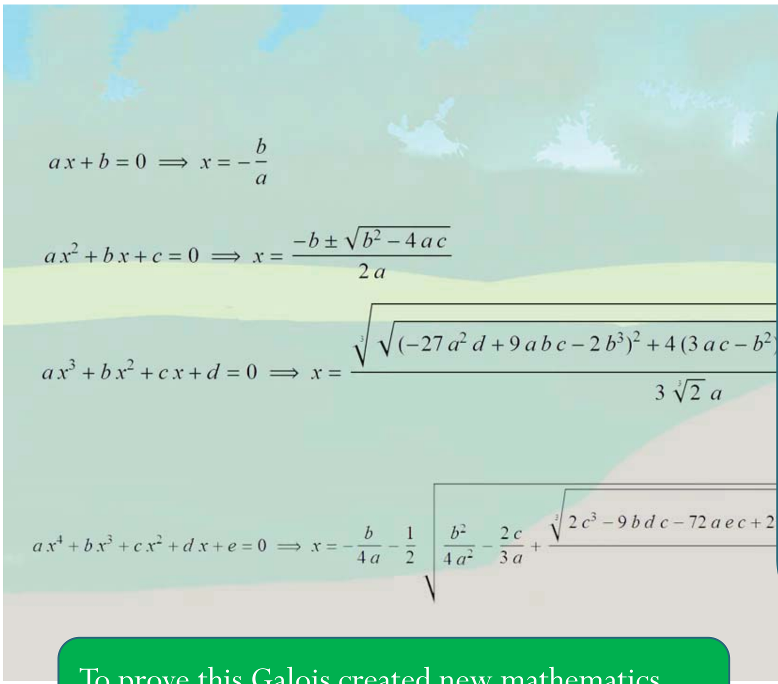
$$a x^2 + b x + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}$$

$$a x^3 + b x^2 + c x + d = 0 \implies x = \frac{\sqrt[3]{\sqrt{(-27 a^2 d + 9 a b c - 2 b^3)^2 + 4(3 a c - b^2)^3} - 27 a^2 d + 9 a b c - 2 b^3}}{3 \sqrt[3]{2}\, a} - \frac{\ldots}{3 a \sqrt[3]{\sqrt{(-27 a^2 \ldots}}}$$

$$a x^4 + b x^3 + c x^2 + d x + e = 0 \implies x = -\frac{b}{4 a} - \frac{1}{2}\sqrt{\frac{b^2}{4 a^2} - \frac{2 c}{3 a} + \frac{\sqrt[3]{2 c^3 - 9 b d c - 72 a e c + 27 a d^2 + 27 b^2 e + \sqrt{(2 c^3 - 9 b d c - 72 a e c + 27 a d^2 + \ldots}}}{3 \sqrt[3]{2}\, a}}$$

[https://www.youtube.com/watch?v=Mc0bvea6G3I]

Évariste Galois: Contribution



In algebra, you learn to solve equations. To solve quadratic equations you use a quadratic formula. To solve cubic equations, you use the less well-known cubic formula and to solve equations of degree four, you use the quartic formula…
Galois proved that for degrees five and higher, there are no general formulas.

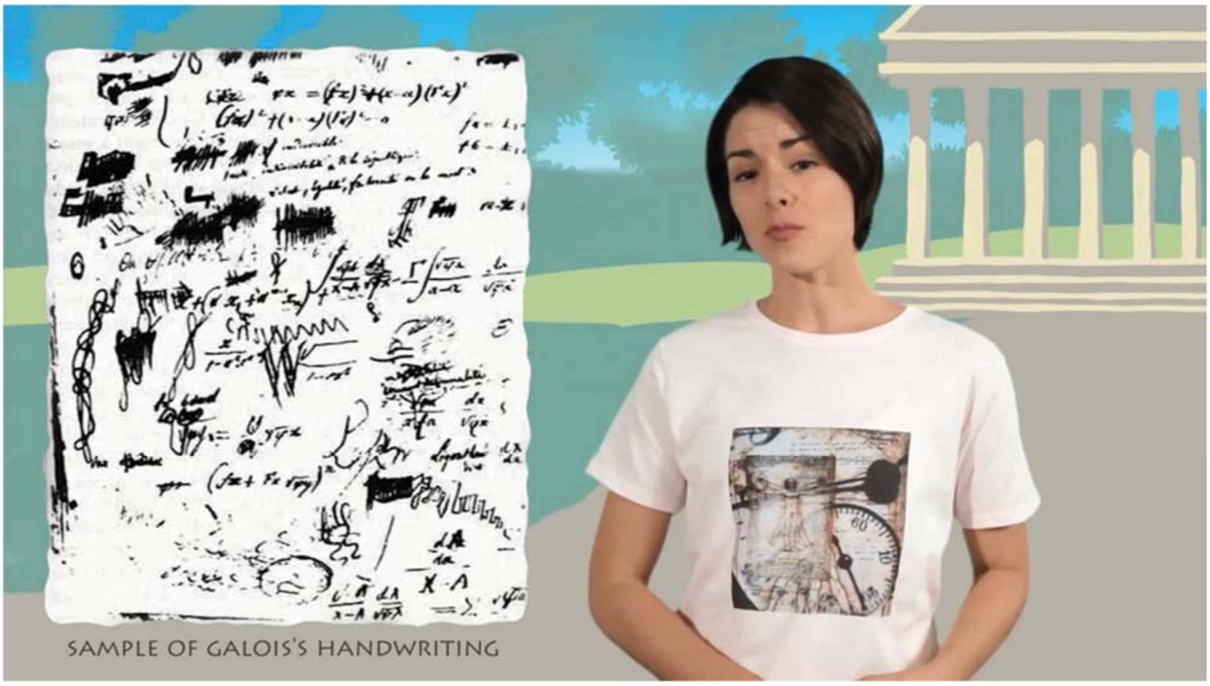
To prove this Galois created new mathematics which we now call **Galois theory** in his honor.

Évariste Galois: Life

[https://www.youtube.com/watch?v=Mc0bvea6G3I]

Évariste Galois: Life

Évariste Galois: Life



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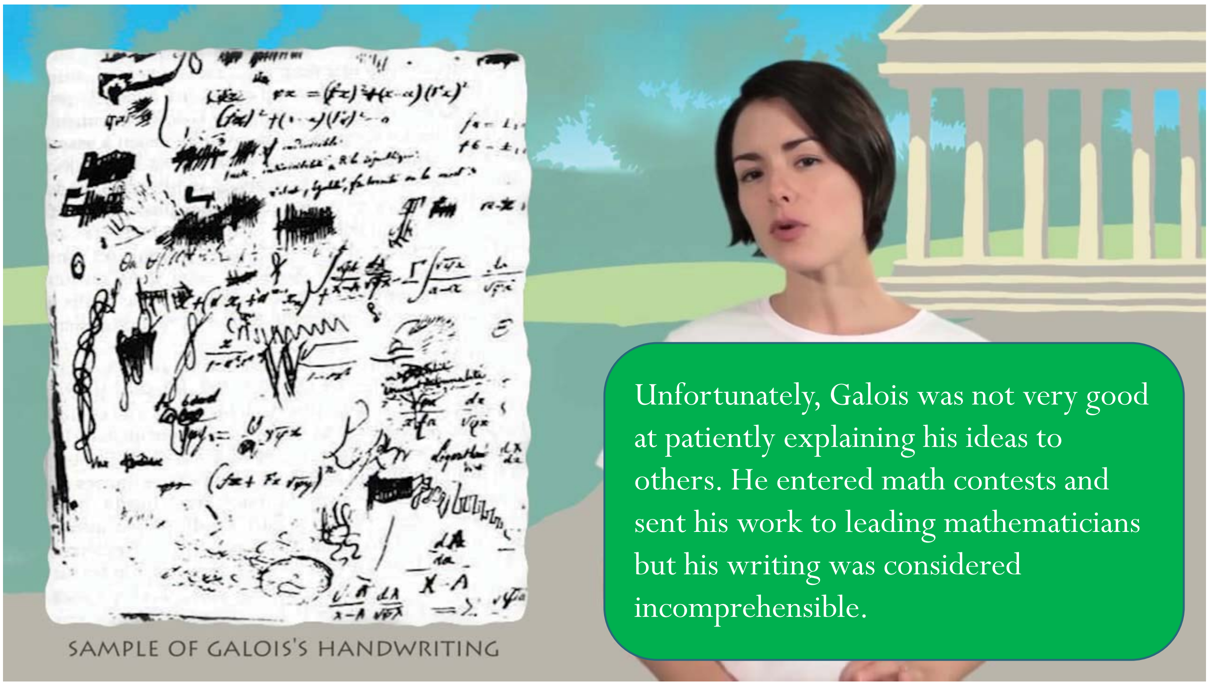
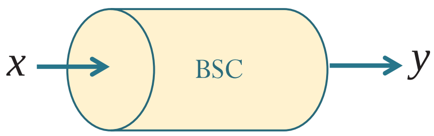
Évariste Galois: Life



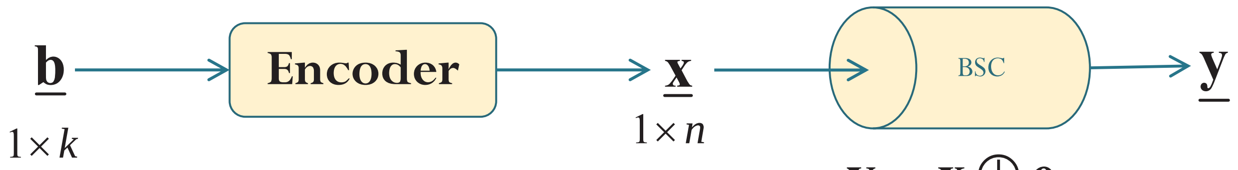
Évariste Galois: Life

BSC and the Error Pattern

- For one use of the channel,



- Again, to transmit k information bits, the channel is used n times.



$$\underline{\mathbf{y}} = \underline{\mathbf{x}} \oplus \underline{\mathbf{e}}$$

error pattern

Its nonzero elements mark the positions of transmission error in $\underline{\mathbf{y}}$

Additional Properties in GF(2)

- The following statements are equivalent
  1. $a \oplus b = c$
  2. $a \oplus c = b$
  3. $b \oplus c = a$

  Having one of these is the same as having all three of them.

- The following statements are equivalent
  1. $\underline{\mathbf{a}} \oplus \underline{\mathbf{b}} = \underline{\mathbf{c}}$
  2. $\underline{\mathbf{a}} \oplus \underline{\mathbf{c}} = \underline{\mathbf{b}}$
  3. $\underline{\mathbf{b}} \oplus \underline{\mathbf{c}} = \underline{\mathbf{a}}$

  Having one of these is the same as having all three of them.

- In particular, because $\underline{\mathbf{x}} \oplus \underline{\mathbf{e}} = \underline{\boldsymbol{y}}$, if we are given two quantities, we can find the third quantity by summing the other two.

Linear Block Codes

- Definition: $\mathcal{C}$ is a **(binary) linear (block) code** if and only if $\mathcal{C}$ forms a vector (sub)space (over GF(2)).

In case you forgot about the concept of vector space,…

  - Equivalently, this is the same as requiring that

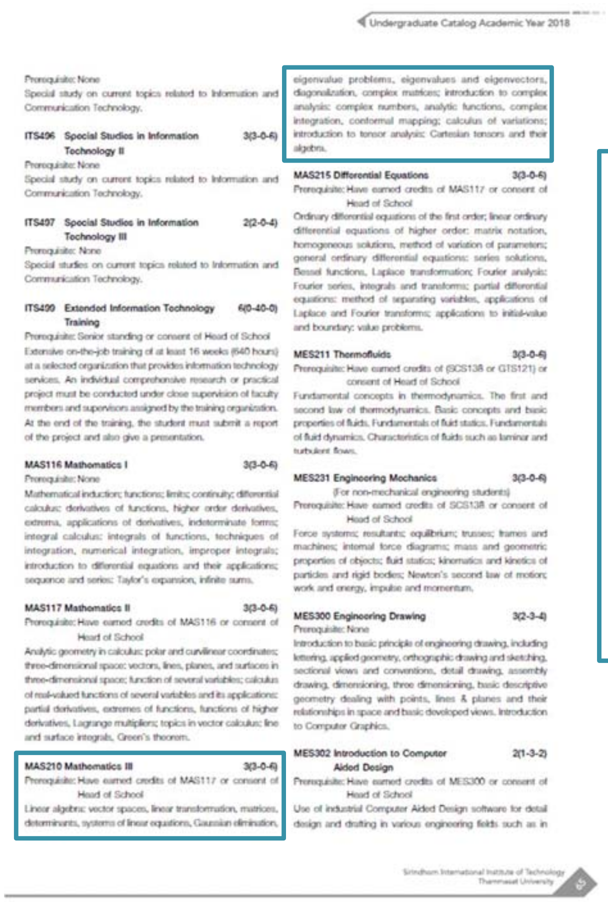
$$\text{if } \underline{\mathbf{x}}^{(1)} \text{ and } \underline{\mathbf{x}}^{(2)} \in \mathcal{C}, \text{ then } \underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} \in \mathcal{C}.$$

  - Note that any (non-empty) linear code $\mathcal{C}$ must contain $\underline{\mathbf{0}}$.

- Ex. The code that we considered in **Problem 5 of HW4** is

$$\mathcal{C} = \{00000, 01000, 10001, 11111\}$$

Is it a linear code?



Undergraduate Catalog Academic Year 2018

Prerequisite: None
Special study on current topics related to Information and Communication Technology.

ITS496 Special Studies in Information Technology II 3(3-0-6)
Prerequisite: None
Special study on current topics related to Information and Communication Technology.

ITS497 Special Studies in Information Technology III 2(2-0-4)
Prerequisite: None
Special studies on current topics related to Information and Communication Technology.

ITS499 Extended Information Technology Training 6(0-40-0)
Prerequisite: Senior standing or consent of Head of School
Extensive on-the-job training of at least 16 weeks (640 hours) at a selected organization that provides information technology services. An individual comprehensive research or practical project must be conducted under close supervision of faculty members and supervisors assigned by the training organization. At the end of the training, the student must submit a report of the project and also give a presentation.

MAS116 Mathematics I 3(3-0-6)
Prerequisite: None
Mathematical induction; functions; limits; continuity; differential calculus: derivatives of functions, higher order derivatives, extrema, applications of derivatives, indeterminate forms; integral calculus: integrals of functions, techniques of integration, numerical integration, improper integrals; introduction to differential equations and their applications; sequence and series: Taylor's expansion, infinite sums.

MAS117 Mathematics II 3(3-0-6)
Prerequisite: Have earned credits of MAS116 or consent of Head of School
Analytic geometry in calculus: polar and curvilinear coordinates; three-dimensional space: vectors, lines, planes, and surfaces in three-dimensional space; function of several variables; calculus of real-valued functions of several variables and its applications: partial derivatives, extremes of functions, functions of higher derivatives, Lagrange multipliers; topics in vector calculus: line and surface integrals, Green's theorem.

MAS210 Mathematics III 3(3-0-6)
Prerequisite: Have earned credits of MAS117 or consent of Head of School
Linear algebra: vector spaces, linear transformation, matrices, determinants, systems of linear equations, Gaussian elimination, eigenvalue problems, eigenvalues and eigenvectors, diagonalization, complex matrices; introduction to complex analysis: complex numbers, analytic functions, complex integration, conformal mapping; calculus of variations; introduction to tensor analysis: Cartesian tensors and their algebra.

MAS215 Differential Equations 3(3-0-6)
Prerequisite: Have earned credits of MAS117 or consent of Head of School
Ordinary differential equations of the first order; linear ordinary differential equations of higher order: matrix notation, homogeneous solutions, method of variation of parameters; general ordinary differential equations: series solutions, Bessel functions, Laplace transformation; Fourier analysis: Fourier series, integrals and transforms; partial differential equations: method of separating variables, applications of Laplace and Fourier transforms; applications to initial-value and boundary value problems.

MES211 Thermofluids 3(3-0-6)
Prerequisite: Have earned credits of (SCS138 or GTS121) or consent of Head of School
Fundamental concepts in thermodynamics. The first and second law of thermodynamics. Basic concepts and basic properties of fluids. Fundamentals of fluid statics. Fundamentals of fluid dynamics. Characteristics of fluids such as laminar and turbulent flows.

MES231 Engineering Mechanics 3(3-0-6)
(For non-mechanical engineering students)
Prerequisite: Have earned credits of SCS138 or consent of Head of School
Force systems; resultants; equilibrium; trusses; frames and machines; internal force diagrams; mass and geometric properties of objects; fluid statics; kinematics and kinetics of particles and rigid bodies; Newton's second law of motion; work and energy, impulse and momentum.

MES300 Engineering Drawing 3(2-3-4)
Prerequisite: None
Introduction to basic principle of engineering drawing, including lettering, applied geometry, orthographic drawing and sketching, sectional views and conventions, detail drawing, assembly drawing, dimensioning, three dimensioning, basic descriptive geometry dealing with points, lines & planes and their relationships in space and basic developed views. Introduction to Computer Graphics.

MES302 Introduction to Computer Aided Design 2(1-3-2)
Prerequisite: Have earned credits of MES300 or consent of Head of School
Use of industrial Computer Aided Design software for detail design and drafting in various engineering fields such as in

Sirindhorn International Institute of Technology
Thammasat University

MAS210 Mathematics III 3(3-0-6)
Prerequisite: Have earned credits of MAS117 or consent of Head of School
Linear algebra: **vector spaces**, linear transformation, **matrices**, determinants, systems of linear equations, Gaussian elimination, eigenvalue problems, eigenvalues and eigenvectors, diagonalization, complex matrices; introduction to complex analysis: complex numbers, analytic functions, complex integration, conformal mapping; calculus of variations; introduction to tensor analysis: Cartesian tensors and their algebra.

Ex. Checking Linearity

- $\mathcal{C} = \{00000, 01000, 10001, 11111\}$
- Step 1: Check that $\mathbf{0} \in \mathcal{C}$.
  - OK for this example.
- Step 2: Check that

$$\text{if } \underline{\mathbf{x}}^{(1)} \text{ and } \underline{\mathbf{x}}^{(2)} \in \mathcal{C}, \text{ then } \underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} \in \mathcal{C}.$$

$\oplus$	00000	01000	10001	11111
00000				
01000				
10001				
11111				

Solution

Ex. Checking Linearity

- $\mathcal{C} = \{00000, 01000, 10001, 11111\}$
- Step 1: Check that $\mathbf{0} \in \mathcal{C}$.
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$$\text{if } \underline{\mathbf{x}}^{(1)} \text{ and } \underline{\mathbf{x}}^{(2)} \in \mathcal{C}, \text{ then } \underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} \in \mathcal{C}.$$

  - Here, we have many counter-examples. So, the code is **not linear**.

$\oplus$	00000	01000	10001	11111
00000	00000	01000	10001	11111
01000	01000	00000	11001	10111
10001	10001	11001	00000	01110
11111	11111	10111	01110	00000

Solution

Checking Linearity

- Step 1: Check that $\mathbf{0} \in \mathcal{C}$.
- Step 2: Check that

  if $\underline{\mathbf{x}}^{(1)}$ and $\underline{\mathbf{x}}^{(2)} \in \mathcal{C}$, then $\underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} \in \mathcal{C}$.
  - It may seem that we need to check $|\mathcal{C}|^2$ pairs.
  - Actually, we need to check only $\binom{n-1}{2}$ pairs.

$\oplus$	00000	01000	10001	11111
00000	00000	01000	10001	11111
01000	01000	00000	11001	10111
10001	10001	11001	00000	01110
11111	11111	10111	01110	00000

$\underline{\mathbf{x}} \oplus \underline{\mathbf{0}} = \underline{\mathbf{x}}$

$\underline{\mathbf{x}} \oplus \underline{\mathbf{x}} = \underline{\mathbf{0}}$

$\underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} = \underline{\mathbf{x}}^{(2)} \oplus \underline{\mathbf{x}}^{(1)}$

Ex. Creating Linearity

- We have checked that

  $\mathcal{C} = \{00000, 01000, 10001, 11111\}$

  is not linear.
- Change one codeword in $\mathcal{C}$ to make the code linear.

$\oplus$	00000			
00000				

Solution

Ex. Creating Linearity

- We have checked that
  $\mathcal{C} = \{00000, 01000, 10001, \overset{11001}{\cancel{11111}}\}$
  is not linear.
- Change one codeword in $\mathcal{C}$ to make the code linear.

For linearity, we always need **0**

If we want these two to be in our code, then their sum must be in our code too. So, we change 11111 to 11001.

⊕	00000	01000	10001	11001
00000				
01000			11001	10001
10001				01000
11111				

Solution

Ex. Creating Linearity

- We have checked that
  $\mathcal{C} = \{00000, 01000, 10001, 11111\}$
  is not linear.
- Change one codeword in $\mathcal{C}$ to make the code linear.
- Three solutions:
  - $\mathcal{C} = \{00000, 01000, 10001, \overset{11001}{\cancel{11111}}\}$
  - $\mathcal{C} = \{00000, 01000, \overset{10111}{\cancel{10001}}, 11111\}$
  - $\mathcal{C} = \{00000, \overset{01110}{\cancel{01000}}, 10001, 11111\}$

⊕	00000	01000	10001	11111
00000	00000	01000	10001	11111
01000	01000	00000	11001	10111
10001	10001	11001	00000	01110
11111	11111	10111	01110	00000

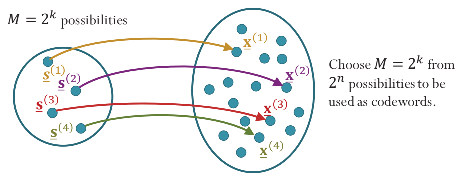
Linear Block Codes: Motivation (1)

- Why linear block codes are popular?
- Recall: General block **encoding**
  - Characterized by its codebook.

[See Section 3.5 of the lecture notes.]

- The table that lists all the 2^k mapping from the k-bit info-block $\underline{\mathbf{s}}$ to the n-bit codeword $\underline{\mathbf{x}}$ is called the **codebook**.
- The M info-blocks are denoted by $\underline{\mathbf{s}}^{(1)}, \underline{\mathbf{s}}^{(2)}, \ldots, \underline{\mathbf{s}}^{(M)}$. The corresponding M codewords are denoted by $\underline{\mathbf{x}}^{(1)}, \underline{\mathbf{x}}^{(2)}, \ldots, \underline{\mathbf{x}}^{(M)}$, respectively.

index i	info-block $\underline{\mathbf{s}}$	codeword $\underline{\mathbf{x}}$
1	$\underline{\mathbf{s}}^{(1)} = 000\ldots0$	$\underline{\mathbf{x}}^{(1)} =$
2	$\underline{\mathbf{s}}^{(2)} = 000\ldots1$	$\underline{\mathbf{x}}^{(2)} =$
$\vdots$	$\vdots$	$\vdots$
M	$\underline{\mathbf{s}}^{(M)} = 111\ldots1$	$\underline{\mathbf{x}}^{(M)} =$



  - Can be realized by combinational/combinatorial circuit.
    - If lucky, can used K-map to simplify the circuit.

Linear Block Codes: Motivation (2)

- Why linear block codes are popular?
- Linear block encoding is the same as matrix multiplication.
  - See next slide.
  - The matrix replaces the table for the codebook.
  - The size of the matrix is only $k \times n$ bits.
    - Compare this against the table (codebook) of size $2^k \times (k+n)$ bits for general block encoding.
- Linearity $\Rightarrow$ easier implementation and analysis
- Performance of the class of linear block codes is similar to performance of the general class of block codes.
  - Can limit our study to the subclass of linear block codes without sacrificing system performance.

Example

- $\mathcal{C} = \{00000, 01000, 10001, 11001\}$
- Let

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Find $\underline{\mathbf{b}}\mathbf{G}$ when $\underline{\mathbf{b}} = [0\ 0]$.
- Find $\underline{\mathbf{b}}\mathbf{G}$ when $\underline{\mathbf{b}} = [0\ 1]$.
- Find $\underline{\mathbf{b}}\mathbf{G}$ when $\underline{\mathbf{b}} = [1\ 0]$.
- Find $\underline{\mathbf{b}}\mathbf{G}$ when $\underline{\mathbf{b}} = [1\ 1]$.

All possible two-bit vectors

Block Matrices

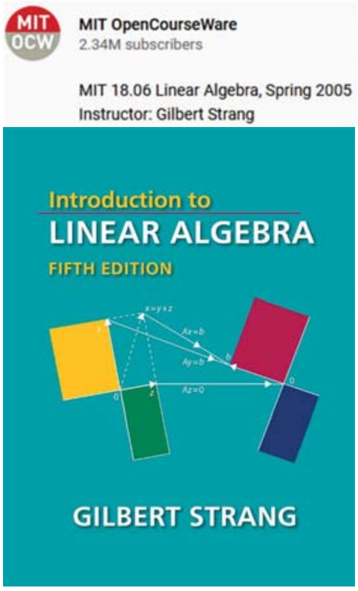
- A **block matrix** or a **partitioned matrix** is a matrix that is interpreted as having been broken into sections called **blocks** or **submatrices**.
- Examples:

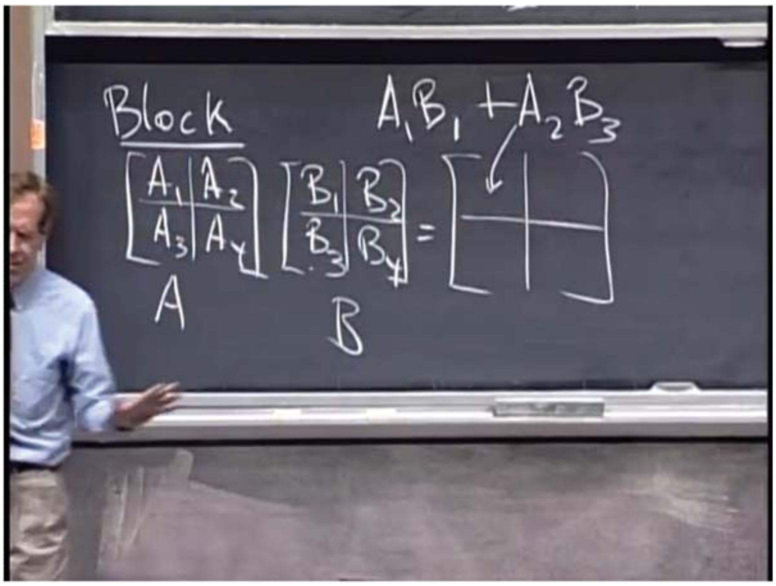
$$\left(\underbrace{\begin{matrix} 10 & 6 \\ 9 & 7 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 6 & 4 & 3 \\ 3 & 5 & 9 \end{matrix}}_{B}\right)$$

$$\begin{pmatrix} \underbrace{\begin{matrix} 2 & 2 & 5 \\ 3 & 3 & 4 \end{matrix}}_{C} & \underbrace{\begin{matrix} 10 & 2 & 10 & 2 & 5 \\ 5 & 10 & 5 & 3 & 6 \end{matrix}}_{D} \\ \underbrace{\begin{matrix} 3 & 3 & 4 \\ 7 & 2 & 5 \\ 8 & 3 & 6 \end{matrix}}_{E} & \underbrace{\begin{matrix} 1 & 1 & 5 & 5 & 6 \\ 3 & 10 & 6 & 10 & 3 \\ 9 & 8 & 3 & 6 & 5 \end{matrix}}_{F} \end{pmatrix}$$

Block Matrix Multiplications

- Matrix multiplication can also be carried out blockwise (assuming that the block sizes are compatible).





[https://youtu.be/FX4C-JpTFgY?t=1103]

Ex: Block Matrix Multiplications

$$\left(\underbrace{\begin{matrix}10 & 6\\ 9 & 7\end{matrix}}_{A}\;\underbrace{\begin{matrix}6 & 4 & 3\\ 3 & 5 & 9\end{matrix}}_{B}\right)\times\left(\begin{matrix}\underbrace{\begin{matrix}2 & 2 & 5\\ 3 & 3 & 4\end{matrix}}_{C} & \underbrace{\begin{matrix}10 & 2 & 10 & 2 & 5\\ 5 & 10 & 5 & 3 & 6\end{matrix}}_{D}\\ \underbrace{\begin{matrix}3 & 3 & 4\\ 7 & 2 & 5\\ 8 & 3 & 6\end{matrix}}_{E} & \underbrace{\begin{matrix}1 & 1 & 5 & 5 & 6\\ 3 & 10 & 6 & 10 & 3\\ 9 & 8 & 3 & 6 & 5\end{matrix}}_{F}\end{matrix}\right)$$

$$=\left(\underbrace{\begin{matrix}108 & 73 & 136\\ 155 & 85 & 164\end{matrix}}_{AC+BE}\;\underbrace{\begin{matrix}175 & 150 & 193 & 126 & 149\\ 224 & 213 & 197 & 158 & 165\end{matrix}}_{AD+BF}\right)$$

$$\left(\underbrace{\begin{matrix}10 & 6 & 6 & 4 & 3\\ 9 & 7 & 3 & 5 & 9\end{matrix}}_{X}\right)\times\left(\underbrace{\begin{matrix}2 & 2 & 5 & 10\\ 3 & 3 & 4 & 5\\ 3 & 3 & 4 & 1\\ 7 & 2 & 5 & 3\\ 8 & 3 & 6 & 9\end{matrix}}_{G}\;\underbrace{\begin{matrix}2 & 10 & 2 & 5\\ 10 & 5 & 3 & 6\\ 1 & 5 & 5 & 6\\ 10 & 6 & 10 & 3\\ 8 & 3 & 6 & 5\end{matrix}}_{H}\right)$$

$$=\left(\underbrace{\begin{matrix}108 & 73 & 136 & 175\\ 155 & 85 & 164 & 224\end{matrix}}_{XG}\;\underbrace{\begin{matrix}150 & 193 & 126 & 149\\ 213 & 197 & 158 & 165\end{matrix}}_{XH}\right)$$

Linear Block Codes: Generator Matrix

For any linear code, there is a matrix $\mathbf{G} = \begin{bmatrix} \boxed{\underline{\mathbf{g}}^{(1)}} \\ \boxed{\underline{\mathbf{g}}^{(2)}} \\ \vdots \\ \boxed{\underline{\mathbf{g}}^{(k)}} \end{bmatrix}_{k \times n}$

called the **generator matrix**
such that, for any codeword $\underline{\mathbf{x}}$, there is a message vector $\underline{\mathbf{b}}$ which produces $\underline{\mathbf{x}}$ by

$$\boxed{\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G}}$$

From $\underline{\mathbf{b}}$ to $\underline{\mathbf{x}}$

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = [b_1 \ \ b_2 \ \cdots \ \ b_k] \begin{bmatrix} \boxed{\underline{\mathbf{g}}^{(1)}} \\ \boxed{\underline{\mathbf{g}}^{(2)}} \\ \vdots \\ \boxed{\underline{\mathbf{g}}^{(k)}} \end{bmatrix}_{k \times n}$$

$$= b_1 \underline{\mathbf{g}}^{(1)} \oplus b_2 \underline{\mathbf{g}}^{(2)} \oplus \cdots \oplus b_k \underline{\mathbf{g}}^{(k)} = \sum_{j=1}^{k} b_j \underline{\mathbf{g}}^{(j)}$$

- Any codeword is simply a linear combination of the rows of **G**.
  - The weights are given by the bits in the message $\underline{\mathbf{b}}$

Linear Combination in GF(2)

- A **linear combination** is an expression constructed from a set of terms by multiplying each term by a constant (weight) and adding the results.
- For example, a linear combination of x and y would be any expression of the form $ax + by$, where a and b are constants.
- General expression:
$$c_1\underline{\mathbf{a}}^{(1)} + c_2\underline{\mathbf{a}}^{(2)} + \cdots + c_k\underline{\mathbf{a}}^{(k)}$$
- In GF(2), c_i is limited to being 0 or 1. So, a linear combination is simply a sum of a sub-collection of the vectors.

Linear Block Codes: Generator Matrix

For any linear code, there is a matrix $\mathbf{G} = \begin{bmatrix} \boxed{\underline{\mathbf{g}}^{(1)}} \\ \boxed{\underline{\mathbf{g}}^{(2)}} \\ \vdots \\ \boxed{\underline{\mathbf{g}}^{(k)}} \end{bmatrix}_{k \times n}$

called the **generator matrix**

such that, for any codeword $\underline{\mathbf{x}}$, there is a message vector $\underline{\mathbf{b}}$ which produces $\underline{\mathbf{x}}$ by

$$\boxed{\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G}} = \underbrace{\sum_{j=1}^{k} b_j \underline{\mathbf{g}}^{(j)}}$$

mod-2 summation

Note:

(1) Any codeword can be expressed as a linear combination of the rows of $\mathbf{G}$

(2) $\mathcal{C} = \{\underline{\mathbf{b}}\mathbf{G} : \underline{\mathbf{b}} \in \{0,1\}^k\}$

Note also that, given a matrix $\mathbf{G}$, the (block) code that is constructed by (2) is always linear.

Fact: If a code is generated by plugging in every possible $\underline{\mathbf{b}}$ into $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G}$, then the code will automatically be linear.

Proof

If $\mathbf{G}$ has k rows. Then, $\underline{\mathbf{b}}$ will have k bits. We can list them all as $\underline{\mathbf{b}}^{(1)}, \underline{\mathbf{b}}^{(2)}, \ldots, \underline{\mathbf{b}}^{(2^k)}$. The corresponding codewords are

$$\underline{\mathbf{x}}^{(i)} = \underline{\mathbf{b}}^{(i)}\mathbf{G} \text{ for } i = 1, 2, \ldots, 2^k.$$

Let's take two codewords, say, $\underline{\mathbf{x}}^{(i_1)}$ and $\underline{\mathbf{x}}^{(i_2)}$. By construction, $\underline{\mathbf{x}}^{(i_1)} = \underline{\mathbf{b}}^{(i_1)}\mathbf{G}$ and $\underline{\mathbf{x}}^{(i_2)} = \underline{\mathbf{b}}^{(i_2)}\mathbf{G}$. Now, consider the sum of these two codewords:

$$\underline{\mathbf{x}}^{(i_1)} \oplus \underline{\mathbf{x}}^{(i_2)} = \underline{\mathbf{b}}^{(i_1)}\mathbf{G} \oplus \underline{\mathbf{b}}^{(i_2)}\mathbf{G} = \left(\underline{\mathbf{b}}^{(i_1)} \oplus \underline{\mathbf{b}}^{(i_2)}\right)\mathbf{G}$$

Note that because we plug in ***every possible*** $\underline{\mathbf{b}}$ to create this code, we know that $\underline{\mathbf{b}}^{(i_1)} \oplus \underline{\mathbf{b}}^{(i_2)}$ should be one of these $\underline{\mathbf{b}}$. Let's suppose $\underline{\mathbf{b}}^{(i_1)} \oplus \underline{\mathbf{b}}^{(i_2)} = \underline{\mathbf{b}}^{(i_3)}$ for some $\underline{\mathbf{b}}^{(i_3)}$. This means

$$\underline{\mathbf{x}}^{(i_1)} \oplus \underline{\mathbf{x}}^{(i_2)} = \underline{\mathbf{b}}^{(i_3)}\mathbf{G}.$$

But, again, by construction, $\underline{\mathbf{b}}^{(i_3)}\mathbf{G}$ gives a codeword $\underline{\mathbf{x}}^{(i_3)}$ in this code. Because the sum of any two codewords is still a codeword, we conclude that the code is linear.

Linear Block Code: Example

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

- Find the codeword for the message $\underline{\mathbf{b}} = [1\ 0\ 0]$
- Find the codeword for the message $\underline{\mathbf{b}} = [0\ 1\ 1]$
- How many codewords do this code have?

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5.1 Binary Linear Block Codes

Review: Linear Block Code and Generator Matrix

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5.1 Binary Linear Block Codes

Generator Matrix, Codebook, and Repetition Code

Linear Block Code: Codebook

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = (b_1\ b_2\ b_3)\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$
$$= (b_1, b_2, b_3, b_1 \oplus b_3, b_2 \oplus b_3, b_1 \oplus b_2)$$

$\underline{\mathbf{b}}$			$\underline{\mathbf{x}}$					
0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	1	1	0
0	1	0	0	1	0	0	1	1
0	1	1	0	1	1	1	0	1
1	0	0	1	0	0	1	0	1
1	0	1	1	0	1	0	1	1
1	1	0	1	1	0	1	1	0
1	1	1	1	1	1	0	0	0

MATLAB: Codebook

```
G = [1 0 0 1 0 1; 0 1 0 0 1 1; 0 0 1 1 1 0];
[B C] = blockCodebook(G)
```

```
function [B C] = blockCodebook(G)
[k n] = size(G);
% All data words
B = dec2bin(0:2^k-1)-'0';
% All codewords
C = mod(B*G,2);
end
```

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\underline{\mathbf{b}}$			$\underline{\mathbf{x}}$					
0	0	0	0	0	0	0	0	0
0	0	1	0	0	1	1	1	0
0	1	0	0	1	0	0	1	1
0	1	1	0	1	1	1	0	1
1	0	0	1	0	0	1	0	1
1	0	1	1	0	1	0	1	1
1	1	0	1	1	0	1	1	0
1	1	1	1	1	1	0	0	0

Linear Block Code: Example

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

- Find the codeword for the message $\underline{\mathbf{b}} = [1\ 0\ 0\ 0]$
- Find the codeword for the message $\underline{\mathbf{b}} = [0\ 1\ 1\ 0]$
- How many codewords do this code have?

MATLAB: Codebook

```
G = [1 1 1 0 0 0 0; 1 0 0 1 1 0 0; 0 0 1 0 1 1 0; 1 0 1 0 1 0 1];
[B C] = blockCodebook(G)
```

```
function [B C] = blockCodebook(G)
[k n] = size(G);
% All data words
B = dec2bin(0:2^k-1)-'0';
% All codewords
C = mod(B*G,2);
end
```

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$\underline{\mathbf{b}}$	$\underline{\mathbf{x}}$
0 0 0 0	0 0 0 0 0 0 0
0 0 0 1	1 0 1 0 1 0 1
0 0 1 0	0 0 1 0 1 1 0
0 0 1 1	1 0 0 0 0 1 1
0 1 0 0	1 0 0 1 1 0 0
0 1 0 1	0 0 1 1 0 0 1
0 1 1 0	1 0 1 1 0 1 0
0 1 1 1	0 0 0 1 1 1 1
1 0 0 0	1 1 1 0 0 0 0
1 0 0 1	0 1 0 0 1 0 1
1 0 1 0	1 1 0 0 1 1 0
1 0 1 1	0 1 1 0 0 1 1
1 1 0 0	0 1 1 1 1 0 0
1 1 0 1	1 1 0 1 0 0 1
1 1 1 0	0 1 0 1 0 1 0
1 1 1 1	1 1 1 1 1 1 1

Summary

Review: Linear Block Codes

- Given a list of codewords for a code $\mathcal{C}$, we can determine whether $\mathcal{C}$ is linear by
  - Definition: if $\underline{\mathbf{x}}^{(1)}$ and $\underline{\mathbf{x}}^{(2)} \in \mathcal{C}$, then $\underline{\mathbf{x}}^{(1)} \oplus \underline{\mathbf{x}}^{(2)} \in \mathcal{C}$
  - Shortcut:
    - First check that $\mathcal{C}$ must contain $\underline{\mathbf{0}}$.
    - Then, check only pairs of the non-zero codewords.
      - One check = three checks
- Codewords can be generated by a **generator matrix**
  - $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = \sum_{i=1}^{k} b_i \underline{\mathbf{g}}^{(i)}$ where $\underline{\mathbf{g}}^{(i)}$ is the i^{th} row of $\mathbf{G}$
- Codebook can be generated by
  - working **row-wise**: generating each codeword one-by-one, or
  - working **column-wise**: first, reading, from $\mathbf{G}$, how each bit in the codeword is created from the bits in $\underline{\mathbf{b}}$; then, in the codebook, carry out the operations on columns of $\underline{\mathbf{b}}$.

Linear Block Codes: Examples

- **Repetition code**: $\underline{\mathbf{x}} = [b \quad b \quad \cdots \quad b]$
  - $\mathbf{G} = [1 \quad 1 \quad \cdots \quad 1]$
  - $\underline{\mathbf{x}} = \underline{\mathbf{b}}\mathbf{G} = b\mathbf{G} = [b \quad b \quad \cdots \quad b]$
  - $R = \frac{k}{n} = \frac{1}{n}$

b	$\underline{\mathbf{x}}$		
0	0	0	0
1	1	1	1

- **Single-parity-check code**: $\underline{\mathbf{x}} = \left[\boxed{\quad \underline{\mathbf{b}} \quad} ; \underbrace{\sum_{j=1}^{k} b_j}_{\text{parity bit}} \right]$
  - $\mathbf{G} = [\mathbf{I}_{k \times k}; \underline{\mathbf{1}}^T]$
  - $R = \frac{k}{n} = \frac{k}{k+1}$

$\underline{\mathbf{b}}$		$\underline{\mathbf{x}}$		
0	0	0	0	0
0	1	0	1	1
1	0	1	0	1
1	1	1	1	0

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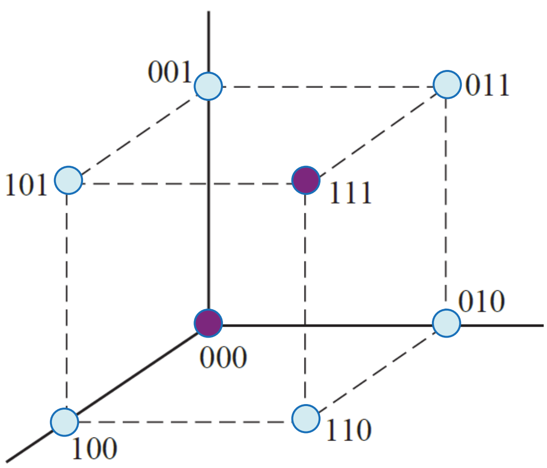
5.1 Binary Linear Block Codes

Single-Parity-Check Code, Parity, and Introduction to Error Detection

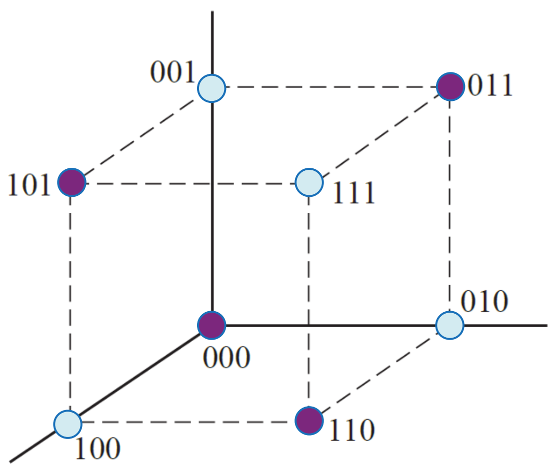
Vectors representing 3-bit codewords

Representing the codewords in the two examples on the previous slide as vectors:



Triple-repetition code

$$P(\mathcal{E}) = 1-(1-p)^3 - 3p(1-p)^2$$

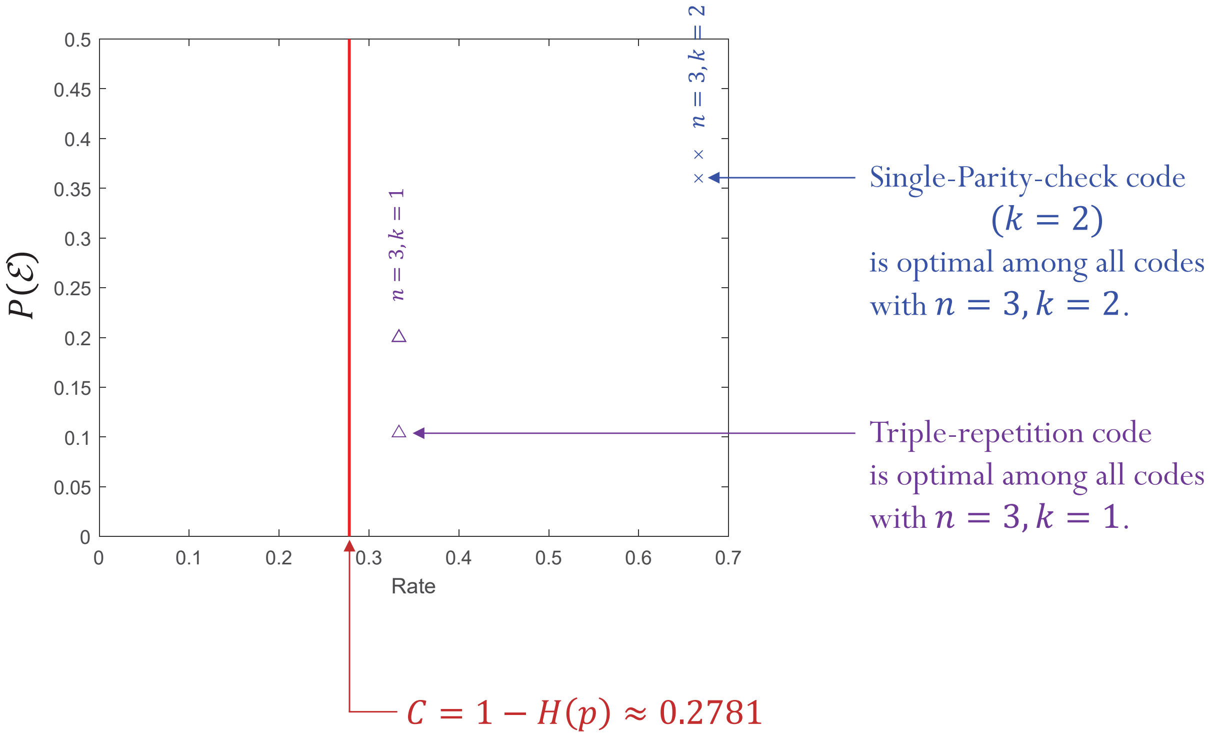


Single-Parity-check code

$$P(\mathcal{E}) = 1-(1-p)^3 - p(1-p)^2$$

Achievable Performance

BSC with $p = 0.2$



Related Idea:

Even Parity vs. Odd Parity

- Parity bit checking is used occasionally for transmitting ASCII characters, which have 7 bits, leaving the 8th bit as a **parity bit**.
- Two options:
  - **Even Parity**: Added bit ensures an <u>even</u> number of 1s in each codeword.
    - A: 1000001**0**
  - **Odd Parity**: Added bit ensures an <u>odd</u> number of 1s in each codeword.
    - A: 1000001**1**

Even Parity vs. Odd Parity

- Even parity and odd parity are properties of a codeword (a vector), not a bit.
- Note: The generator matrix $\mathbf{G} = [\mathbf{I}_{k\times k}; \underline{\mathbf{1}}^T]$ previously considered produces even parity codeword

$$\underline{\mathbf{x}} = \left[\boxed{\quad\underline{\mathbf{b}}\quad}; \sum_{j=1}^{k} b_j\right]$$

- Q: Consider a code that uses odd parity. Is it linear?

Error Control using Parity Bit

- If an odd number of bits (including the parity bit) are transmitted incorrectly, the parity will be incorrect, thus indicating that a parity error occurred in the transmission.
- Ex.
  - Suppose we use even parity.
  - Consider the codeword $\underline{\mathbf{x}} = 10000010$ 

- Suitable for *detecting* errors; *cannot correct* any errors

US UK

The ASCII Coded Character Set

(American Standard Code for Information Interchange)

Bit Number					6	0	0	0	0	1	1	1	1
					5	0	0	1	1	0	0	1	1
					4	0	1	0	1	0	1	0	1
				Hex	1st	0	1	2	3	4	5	6	7
3	2	1	0	2nd		0	16	32	48	64	80	96	112
0	0	0	0	0		NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1		SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2		STX	DC2	“	2	B	R	b	r
0	0	1	1	3		ETX	DC3	#	3	C	S	c	s
0	1	0	0	4		EOT	DC4	$	4	D	T	d	t
0	1	0	1	5		ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6		ACK	SYN	&	6	F	V	f	v
0	1	1	1	7		BEL	ETB	’	7	G	W	g	w
1	0	0	0	8		BS	CAN	(	8	H	X	h	x
1	0	0	1	9		HT	EM	)	9	I	Y	i	y
1	0	1	0	A		LF	SUB	*	:	J	Z	j	z
1	0	1	1	B		VT	ESC	+	;	K	[	k	{
1	1	0	0	C		FF	FS	,	<	L	\	l	|
1	1	0	1	D		CR	GS	-	=	M	]	m	}
1	1	1	0	E		SO	RS	.	>	N	^	n	~
1	1	1	1	F		SI	US	/	?	O	—	o	DEL

[The ARRL Handbook for Radio Communications 2013]